FINSIM 10 TORSIONAL STIFFNESS OF ROCKET FINS THICKNESS-TAPERED FROM ROOT TO TIP

Technical Note 2021-1 by John Cipolla, July 2021

Rocket fin torsional constant, J and torsional stiffness, K = GJ/b are determined by treating the fin as a tapered beam whose fin thickness, t varies linearly along the span length, b from root to tip. This analysis predicts torsional constant, J decreases significantly as fin thickness is linearly decreased or tapered from fin root to fin tip. Decreasing the torsional constant decreases torsional stiffness thereby lowering fin flutter velocity. Angle of twist, φ for tapered fins necessary to determine fin torsional constant, fin torsional stiffness and flutter velocity is given by the following analysis.

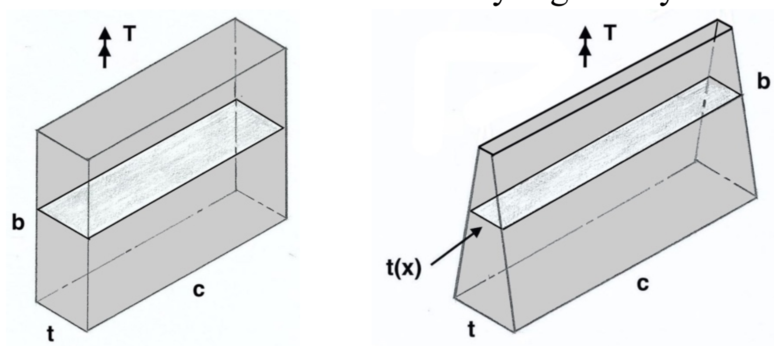

Figure-1, Flat fin geometry (left) compared to tapered fin geometry (right)

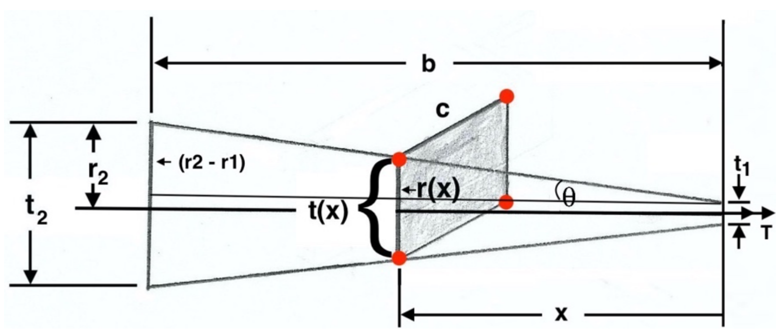

Figure-2, Tapered Fin: [x = 0, t = t_1], [x = b, t = t_2], $\tan\theta = (r_2 - r_1)/b$

The general expression for angle of twist at any section, x for a shaft or fin of linearly elastic material follows from the standard ***Mechanics of Solids*** equation[1].

$$\varphi = \frac{T}{G}\int_0^b \frac{dx}{J(x)} \qquad \text{where } J(x) = \frac{c\, t(x)^3}{3} \tag{1}$$

From the tapered fin geometry expressed in Figure-1 and Figure-2, fin thickness, t as a function of x from the tip of the tapered fin is the following.

$$r = x\, \tan\theta + r_1 \qquad \tan\theta = (r_2 - r_1)/b \tag{2}$$

$$t = 2x\, \tan\theta + 2r_1 \qquad t = \frac{1}{b}(t_2 - t_1)x + t_1 \tag{3}$$

From the generalized torsional constant equation derive the torsional constant for any fin whose thickness varies as a function of x along span, b.

$$J(x) = \frac{c\,t^3}{3} \qquad\qquad J(x) = \frac{c}{3}\left(\frac{1}{b}(t_2 - t_1)x + t_1\right)^3 \tag{4}$$

Substitute the resulting generalized torsional constant, J(x) into the angle of twist equation (Equation-1) to generate the angle of twist for a tapered fin or beam.

$$\varphi = \frac{T}{G}\int_0^b \frac{dx}{J(x)} \qquad\qquad \varphi = \frac{3}{c}\frac{T}{G}\int_0^b \frac{dx}{\left(\frac{1}{b}(t_2-t_1)x+t_1\right)^3} \tag{5}$$

The integral in Equation-5 has been evaluated using the CRC Standard Mathematical Tables[2] and then verified using the symbolic mathematical capability of MathCAD.

$$\int_0^b \frac{dx}{\left(\frac{1}{b}(t_2-t_1)x+t_1\right)^3} = \frac{b}{2{t_1}^2(t_2-t_1)} - \frac{b}{2{t_2}^2(t_2-t_1)} \tag{6}$$

Insert integral, Equation-6 into Equation-5 for tapered fin angle of twist.

$$\varphi = \frac{3}{c}\frac{T}{G}\left[\frac{b}{2{t_1}^2(t_2-t_1)} - \frac{b}{2{t_2}^2(t_2-t_1)}\right] \tag{7}$$

As further validation for angle of twist of constant thickness rectangular cross-section fins, the author consulted *Theory of Elasticity*[5] where on page 277 the following equation for angle of twist of noncircular shafts is presented with coefficients based on the ratio of average fin chord to average fin thickness, where for this example, c/t_{avg} >10 and $\beta = 0.333$.

$$\varphi = \frac{T\,b}{\beta\, c\, {t_{avg}}^3\, G} \tag{7x}$$

Continuing, the torsional constant for tapered fin with torsion, T and length, b.

$$\boldsymbol{J}_\Delta = \frac{Tb}{G\phi} \tag{8}$$

Equivalent torsional stiffness for a tapered fin around the span axis becomes.

$$K_\alpha = \frac{G\,J_\Delta}{b} \tag{9}$$

Radius of gyration[4] (page 5-30) for the tapered cross-section specified in Figure-2.

$$r_\alpha = \sqrt{\frac{c^2 + {t_{avg}}^2}{12}} \text{ where } t_{avg} = \frac{t_1+t_2}{2} \tag{10}$$

Mass for tapered fin specified in Figure-1 and Figure-2.

$$m = \rho_{material}\; c\; b\; t_{avg} \tag{11}$$

Mass moment of inertia for the tapered fin described in Figure-1 and Figure-2.

$$I_\alpha = m{r_\alpha}^2 \tag{12}$$

Finally, torsional frequency[3] of a tapered fin in radians per second (Hz) becomes.

$$\omega_\alpha = \frac{\pi}{2}\sqrt{\frac{K_\alpha}{I_\alpha}} \tag{13}$$

Tapered Circular Shaft Validation and Test
Example 5-12 on page 174, Introduction to Mechanics of Solids

The general expression for angle of twist at any section, x for a shaft of linearly elastic material follows from the standard ***Mechanics of Solids*** equation[1].

$$\varphi = \frac{T}{G}\int_0^L \frac{dx}{J(x)} \qquad \text{where J(x)} = \pi\frac{d(x)^4}{32} \tag{1b}$$

From the tapered fin geometry expressed in Figure-1 and Figure-2, shaft diameter, d as a function of x from the tip of the tapered shaft is the following.

$$r = x\ \tan\theta + r_1 \qquad \tan\theta = (d_2 - d_1)/L \tag{2b}$$

$$d = 2x\ \tan\theta + 2d_1 \qquad d = \frac{1}{L}(d_2 - d_1)x + d_1 \tag{3b}$$

From the generalized torsional constant equation derive the torsional constant for any shaft whose thickness varies as a function of x along length, L.

$$J(x) = \pi\frac{d^4}{32} \qquad J(x) = \frac{\pi}{32}\left(\frac{1}{L}(d_2 - d_1)x + d_1\right)^4 \tag{4b}$$

Substitute the resulting generalized torsional constant, J(x) into the angle of twist equation (Equation-1b) to generate the angle of twist for a tapered shaft.

$$\varphi = \frac{T}{G}\int_0^L \frac{dx}{J(x)} \qquad \varphi = \frac{32}{\pi}\frac{T}{G}\int_0^L \frac{dx}{\left(\frac{1}{L}(d_2-d_1)x+d_1\right)^4} \tag{5b}$$

The integral in Equation-5b was evaluated using the CRC Standard Mathematical Tables[2] and verified using the symbolic mathematical capability of MathCAD.

$$\int_0^L \frac{dx}{\left(\frac{1}{L}(d-d_1)x+d_1\right)^4} = \frac{L}{3{d_1}^3(d_2-d_1)} - \frac{L}{3{d_2}^3(d_2-d_1)} \tag{6b}$$

Insert integral, Equation-6b into Equation-5b for tapered shaft angle of twist.

$$\varphi = \frac{32}{\pi}\frac{T}{G}\left[\frac{L}{3{d_1}^3(d_2-d_1)} - \frac{L}{3{d_2}^3(d_2-d_1)}\right] \tag{7b}$$

Torsional constant for tapered shaft with torsion, T and length, L.

$$\boldsymbol{J}_{\Delta} = \frac{TL}{G\phi} \tag{8b}$$

Equivalent torsional stiffness for a tapered shaft around the length axis becomes.

$$K_{\alpha} = \frac{G\,J_{\Delta}}{L} \tag{9b}$$

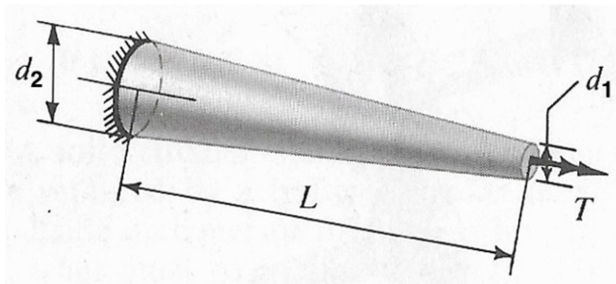

Figure-3, Solid, tapered shaft rigidly fixed at one end

Equation-7b was evaluated using example 5-12 from ***Introduction to Mechanics of Solids*** to determine angular rotation of a solid, tapered steel shaft fastened to a fixed support at one end and subjected to torque T at the other end. This analysis found the angular rotation at the free end when d_1 = 2 in.; d_2 = 6 in.; L = 20 in.; and T = 27,000 in-lb. Using the usual assumptions for strain in prismatic circular shafts subjected to applied torque, T and letting $G = 12\ x\ 10^6\ psi$, equation-7b results for angle of twist, ϕ agree exactly with the answer from the textbook. $Ans.\phi = 0.263^0$.

SUMMARY

Using the derived equations, the torsional constant, J for the tapered shaft example is $9.803\ in^4$ and angle of twist $\phi = 0.263^0$. By comparison the torsional constant, J for a shaft of constant diameter d_2 is $127.235\ in^4$ having angle of twist of only 0.021 degrees for identical torque loading. These results strongly imply the "stiffer" constant diameter shaft has a much greater torsional frequency than a tapered shaft having the same base diameter. Because a constant diameter shaft has greater torsional frequency than a tapered shaft it follows the flutter velocity for a constant diameter shaft is greater than a tapered shaft having the same root diameter.

Final Note: This analysis and a separate analysis using MathCAD indicate that bending frequency, ω_h has a weak effect on torsion-flutter velocity. Because of the weak effect bending frequency has on torsion-flutter velocity the bending frequency of a tapered fin can be taken to be the bending frequency for a flat plate of constant root thickness, t or some average thickness that suits the tapered-fin design.

Flutter Velocity and Torsional Frequency Approach
Constant Thickness Values as Fin Tip Thickness Increased

Figure-4 and Figure-5 plot tapered fin flutter velocity for a typical supersonic rocket determined using Equation-8 to Equation-13 i.e., torsional constant, torsional stiffness, radius of gyration, tapered fin mass, mass moment of inertia and torsional frequency. This analysis illustrates that fin flutter velocity is greatest when fin tip thickness is increased to match the thickness at the base or root of the fin.

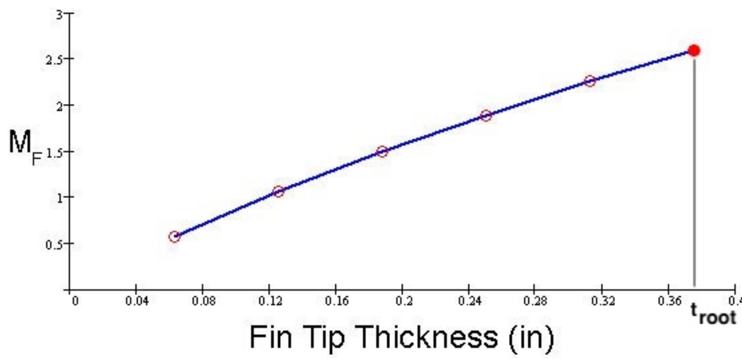

Figure-4, Flutter Mach number verses increasing fin tip thickness

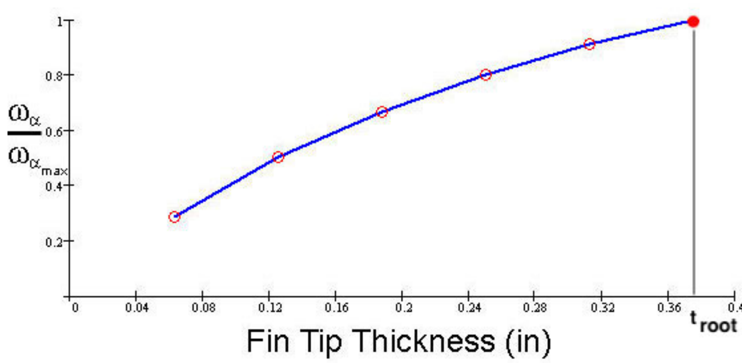

Figure-5, Ratio of tapered fin torsional frequency for fin-tip thickness that increases from 0.0625 inches at the tip to the root thickness specified as 0.375 inches

REFERENCES

[1]Egor P. Popov, *Introduction to Mechanics of Solids* (Prentice-Hall, Inc, New Jersey)
[2]William H. Beyer and Samuel M. Selby*, CRC Standard Mathematical Tables* (CRC Press, Cleveland, Ohio)
[3]Raymond Bisplinghoff, Holt Ashley and Robert Halfman*, Aeroelasticity* (Dover Publications Inc, Mineola, NY)
[4]Avallone and Baumeister*, Marks' Standard Handbook for Mechanical Engineers* (McGraw Hill Inc, NY)
[5]S. Timoshenko and J.N. Goodier*, Theory of Elasticity,* 2nd Edition (McGraw Hill, p. 277, NY)